JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **204**, 124–137 (1996)
ARTICLE NO. 0427

Norm Continuity (for $t > 0$) of Propagators of Arbitrary Order Abstract Differential Equations in Hilbert Spaces*

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Submitted by Joseph A. Ball

Received June 29, 1995

Let (ACP$_n$), $u^{(n)}(t) + \sum_{k=0}^{n-1} A_k u^{(k)}(t) = 0$ $(t \geq 0)$, $u^{(k)}(0) = u_k$ $(0 \leq k \leq n-1)$ be strongly wellposed where the A_k $(0 \leq k \leq n-1)$ are densely defined closed linear operators in a Hilbert space. Let $S_k(t)$ $(0 \leq k \leq n-1)$ be the propagators of (ACP$_n$) and $P_\lambda = \lambda^n I + \sum_{k=0}^{n-1} \lambda^k A_k$. It is shown that $S_k^{(k)}(t)$ $(\forall\, 0 \leq k \leq n-1)$ is norm continuous for $t > 0$ if and only if

$$\lim_{|\omega|\to\infty} \left\| (\tau_0 + i\omega)^{n-1} P_{\tau_0 + i\omega}^{-1} \right\| = 0,$$

$$\lim_{|\omega|\to\infty} \left\| (\tau_0 + i\omega)^{k-1} \overline{P_{\tau_0 + i\omega}^{-1} A_k} \right\| = 0, \qquad 1 \leq k \leq n-1,$$

for a suitable real number τ_0. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let $(H, \langle \cdot , \cdot \rangle)$ be a complex Hilbert space, A_k $(0 \leq k \leq n-1)$ be densely defined closed linear operators in H. Of concern is the abstract

* Research supported by the National NSF of China and the ABSF of Yunnan Province.

0022-247X/96 $18.00

Cauchy problem

$$\begin{cases} u^{(n)}(t) + \sum_{k=0}^{n-1} A_k u^{(k)}(t) = 0, & t \geq 0; \\ u^{(k)}(0) = u_k & (0 \leq k \leq n-1). \end{cases} \tag{ACP$_n$}$$

Problem (ACP$_n$) models many phenomena in nature which vary in time. Initial value problems with time−independent operators are suitable in this form. Mixed problems with time−independent operators and time−independent boundary conditions are also suitable in this form. In the case of mixed problems the boundary conditions are absorbed into the definitions of $D(A_k)$ $(0 \leq k \leq n-1)$ and the fundamental space. Among the prototypes of (ACP$_n$) are linear elastic systems with damping, the damped wave systems, the mathematical model of motion of a thin panel, etc. The study on (ACP$_n$) has attracted many researchers' attention (see, e.g., [1−7, 9−23] and references therein).

Following [24, 25], the authors [26] carried on the study of the strong wellposedness and the analyticity of (ACP$_n$) in the case of Banach spaces, and obtained a Hille−Yosida−Phillips type characterization for (ACP$_n$) to be strongly wellposed, a characterization for analyticity of (ACP$_n$), a general criterion for solvability of (ACP$_n$), and a perturbation theorem (these results were also announced in [27]). In this paper, we pay our attention to the characterization of the norm continuity for $t > 0$ of the propagators of (ACP$_n$).

Recently, stimulated by [18, Theorem 2.3.6, p. 50] and the techniques of [11], P. You [29] obtained successfully a sufficient and necessary condition for the norm continuity for $t > 0$ of a C_0 semigroup in a Hilbert space, in terms of the spectral property of its infinitesimal generator. It is also a characterization of the norm continuity for $t > 0$ of the propagator of a wellposed (ACP$_1$). The purpose of this note is to establish an analogous characterization for the norm continuity for $t > 0$ of the propagators of an arbitrary order strongly wellposed (ACP$_n$). Noting that for the special case of $n = 1$, strong wellposedness is equivalent to wellposedness, we can see that our present result is an extension of [29, Theorem 1].

Throughout this paper, **R** denotes the real numbers, $\mathbf{R}^+$ the nonnegative real numbers, **C** the complex plane, **N** the positive integers, $B(H)$ the set of bounded linear operators from H to H, and

$$P_\lambda = \lambda^n + \sum_{k=0}^{n-1} \lambda^k A_k, \qquad \lambda \in \mathbf{C}.$$

DEFINITION 1 [5]. A solution of (ACP_n) is a function of $u(\cdot) \in C^n(R^+, H)$ such that $u^{(k)}(t) \in D(A_k)$, $A_k u^{(k)}(\cdot) \in C(\mathbf{R}^+, H)$, $0 \le k \le n - 1$, and (ACP_n) is satisfied.

DEFINITION 2 [5]. Problem (ACP_n) is wellposed if and only if

(a) There exist dense subspaces $D_0, \dots, D_{n-1}$ of H such that for $u_0 \in D_0, \dots, u_{n-1} \in D_{n-1}$, (ACP_n) has a solution.

(b) There exists a nondecreasing function $K(t)$ defined in $t \ge 0$ such that $\|u(t)\| \le \sum_{k=0}^{n-1} K(t)\|u^{(k)}(0)\|$ $(t \ge 0)$, for any solution of (ACP_n).

Assuming that (ACP_n) is wellposed, we then have n propagators $S_0, \dots, S_{n-1}$; S_k is defined in D_k by $S_k(t)x = u_k(t)$ $(t \ge 0,\ x \in D_k)$, where $u_k(\cdot)$ is the solution of (ACP_n) with $u_k^{(l)}(0) = \delta_{kl}x$, δ_{kl} the Kronecker delta $(0 \le k, l \le n-1)$, and extended to all of H by continuity.

DEFINITION 3 [26, 27]. Problem (ACP_n) is called strongly wellposed if the hypotheses in [5, Theorem 6.1] hold, i.e., (ACP_n) is wellposed and for each $x \in H$, $S_k(\cdot)x \in C^{(k)}(\mathbf{R}^+, H)$, $S_{n-1}^{(k-1)}(t)x \in D(A_k)$, and $A_k S_{n-1}^{(k-1)}(\cdot)x$ is continuous in $\mathbf{R}^+$, $1 \le k \le n-1$.

In the following discussion, we assume that (ACP_n) is strongly wellposed. By $S_k^{(k)}(\cdot)$ $(0 \le k \le n-1)$, we denote the operators defined as $(S_k^{(k)}(\cdot))x = (S_k(\cdot)x)^{(k)}$ for every $x \in H$. By virtue of the strong wellposedness of (ACP_n), we know from [5] or [26] that there exist positive constants C, $\mu > 0$ such that

$$\left\| S_k^{(k)}(t) \right\| \le Ce^{\mu t}, \qquad t \ge 0, \tag{1.1}$$

$$\sum_{i=k+1}^{n} \lambda^{i-1} \overline{P_\lambda^{-1} A_i} x = \int_0^\infty e^{-\lambda t} S_k^{(k)}(t)\, x\, dt, \tag{1.2}$$

$$x \in H,\ \mathrm{Re}\,\lambda \ge \mu,\ 0 \le k \le n-1,$$

where $A_n = I$ and $\overline{P_\lambda^{-1} A_i}$ is a bounded extension of $P_\lambda^{-1} A_i$ with domain $\bigcap_{l=0}^{i} D(A_l)$ for each $1 \le i \le n-1$. Hence for $x \in H$, $\mathrm{Re}\,\lambda \ge \mu$,

$$\lambda^{n-1} P_\lambda^{-1} x = \int_0^\infty e^{-\lambda t} S_{n-1}^{(n-1)}(t)\, x\, dt, \tag{1.3}$$

$$\lambda^{k-1} \overline{P_\lambda^{-1} A_k} x = \int_0^\infty e^{-\lambda t} \left[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t) \right] x\, dt, \qquad 1 \le k \le n-1. \tag{1.4}$$

2. RESULTS AND PROOFS

LEMMA 1. *Let E be a complex Banach space and let $G(\cdot)$ be an E-valued function defined on* $\mathbf{R}$ *with* $\|G(\cdot)\| \in L_1(\mathbf{R})$. *Then*

$$\left\| \int_{-\infty}^{\infty} e^{ist} G(t)\, dt \right\| \to 0, \qquad \text{as } |s| \to +\infty \ (s \in \mathbf{R}).$$

Proof. Using the arguments similarly as in the scalar case (cf. [8, Proof of (21.39)] and noting [9, Theorem 3.8.3], we can get the conclusion.

LEMMA 2 [28]. *Let $f(t)$ $(t \geq 0)$ be a strongly measurable H-valued function satisfying $\|f(t)\| \leq Ce^{at}$ $(t \geq 0)$ for some C, $a \geq 0$. Then for each* $\tau > a$, $g(\omega) := \| \int_0^\infty e^{-(\tau + i\omega)t} f(t)\, dt \| \in L_2(\mathbf{R})$.

Proof. Proceeding similarly as in the proof of [11, Lemma 1], we can show that $g(\cdot)$ is the Fourier transform of a function in $L_1(\mathbf{R}) \cap L_\infty(\mathbf{R})$; then an application of [8, Lemma (21, 50)] shows the result desired.

THEOREM 1. $S_k^{(k)}(t)$ $(\forall\ 0 \leq k \leq n-1)$ *is norm continuous for $t > 0$ if and only if there is a $\tau_0 > \mu$ such that*

$$\lim_{|\omega| \to \infty} \left\| (\tau_0 + i\omega)^{n-1} P_{\tau_0 + i\omega}^{-1} \right\| = 0, \tag{2.1}$$

$$\lim_{|\omega| \to \infty} \left\| (\tau_0 + i\omega)^{k-1} \overline{P_{\tau_0 + i\omega}^{-1} A_k} \right\| = 0, \qquad 1 \leq k \leq n-1. \tag{2.2}$$

Proof. Necessity. Since $S_{n-1}^{(n-1)}(t)$ is norm continuous for $t > 0$, we have by (1.3),

$$\lambda^{n-1} P_\lambda^{-1} = \int_0^\infty e^{-\lambda t} S_{n-1}^{(n-1)}(t)\, dt, \qquad \operatorname{Re} \lambda \geq \mu. \tag{2.3}$$

We now write (2.3) in the form

$$\lambda^{n-1} P_\lambda^{-1} = \int_{-\infty}^{+\infty} e^{ist} G(t)\, dt, \qquad \operatorname{Re} \lambda \geq \mu, \tag{2.4}$$

where

$$s = -\operatorname{Im} \lambda, \qquad G(t) = \begin{cases} e^{-Re\lambda t} S_{n-1}^{(n-1)}(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Therefore, according to (1.1) and Lemma 1, we obtain (2.1).

Similarly, the hypotheses, together with (1.4), (1.1), and Lemma 1 show that (2.2) holds.

Sufficiency. This proof will be divided into four steps.

First, we prove that

$$J := \Big\{\lambda \in \mathbf{C}\colon P_\lambda^{-1} \text{ exists and belongs to } L(H),\, P_\lambda^{-1}A_k$$
$$\text{is closable and } \overline{P_\lambda^{-1}A_k} \in L(H)\ (1 \le k \le n-1)\Big\} \tag{2.5}$$

is an open subset of $\mathbf{C}$, the $\overline{P_\lambda^{-1}A_k}$ $(1 \le k \le n-1)$ are analytic in J, and

$$\frac{d}{d\lambda}\Big(\overline{P_\lambda^{-1}A_k}\Big) = -\left(\sum_{l=1}^{n} l\lambda^{l-1}\overline{P_\lambda^{-1}A_l}\right)\overline{P_\lambda^{-1}A_k}, \qquad 1 \le k \le n-1. \tag{2.6}$$

For any $\lambda_0 \in J$, $\lambda \in \mathbf{C}$, and $x \in \bigcap_{k=0}^{n-1} D(A_k)$,

$$\begin{aligned} P_\lambda x &= P_{\lambda_0}\left(I + \sum_{l=1}^{n}\big(\lambda^l - \lambda_0^l\big)\overline{P_{\lambda_0}^{-1}A_l}\right)x \\ &= P_{\lambda_0}\big(I + U(\lambda_0, \lambda)\big)x, \end{aligned} \tag{2.7}$$

where $U(\lambda_0, \lambda) = \sum_{l=1}^{n}(\lambda^l - \lambda_0^l)\overline{P_{\lambda_0}^{-1}A_l}$. Since

$$\begin{aligned} U(\lambda_0, \lambda) &= \sum_{l=1}^{n}\Big[(\lambda - \lambda_0 + \lambda_0)^l - \lambda_0^l\Big]\overline{P_{\lambda_0}^{-1}A_l} \\ &= \sum_{l=1}^{n}\left[\sum_{i=1}^{l} C_l^i\lambda_0^{l-i}(\lambda - \lambda_0)^i\right]\overline{P_{\lambda_0}^{-1}A_l} \\ &= \sum_{i=1}^{n}\left[\sum_{l=i}^{n} C_l^i\lambda_0^{l-i}\overline{P_{\lambda_0}^{-1}A_l}\right](\lambda - \lambda_0)^i, \end{aligned} \tag{2.8}$$

we have that for each $\lambda_0 \in J$, there exists a constant $\delta > 0$ such that $\|U(\lambda_0, \lambda)\| \le 1/2$ when $|\lambda - \lambda_0| < \delta$. Therefore, for every $\lambda \in \{\lambda \in C\colon |\lambda - \lambda_0| < \delta\}$, it follows from (2.7) that P_λ^{-1} exists and

$$P_\lambda^{-1} = \big[I + U(\lambda_0, \lambda)\big]^{-1}P_{\lambda_0}^{-1}.$$

This identity implies that J is an open subset of $\mathbf{C}$ and for each $1 \le k \le n-1$, $\overline{P_\lambda^{-1}A_k}$ is analytic in J due to the analyticity of $U(\lambda_0, \lambda)$. Furthermore, from

$$\begin{aligned} P_\lambda^{-1} - P_{\lambda_0}^{-1} &= P_\lambda^{-1}\big(P_{\lambda_0} - P_\lambda\big)P_{\lambda_0}^{-1} \\ &= (\lambda_0 - \lambda)\left[\sum_{l=1}^{n}\sum_{i=0}^{l-1}\lambda_0^i\lambda^{l-i-1}\overline{P_\lambda^{-1}A_l}\right]P_{\lambda_0}^{-1} \qquad (\lambda_0, \lambda \in J), \end{aligned}$$

we know that (2.6) is true.

Secondly, by (2.1) and (2.2), we obtain that for each $1 \le i \le n$,

$$\lim_{|\omega|\to\infty}\sum_{l=i}^{n} C_l^i(\tau_0 + i\omega)^{l-i}\overline{P_{\tau_0+i\omega}^{-1}A_l} = 0.$$

Hence, there exists a nondecreasing sequence $\{\omega_m\}_{m=1}^{\infty}$ with $\omega_m \geq m$ $(m \in \mathbf{N})$ such that for $|\omega| \geq \omega_m$, $m \in \mathbf{N}$,

$$\left\| \sum_{l=i}^{n} C_l^i (\tau_0 + i\omega)^{l-i} \overline{P_{\tau_0 + i\omega}^{-1} A_l} \right\| \leq \frac{1}{2nm^n}, \qquad 1 \leq i \leq n. \tag{2.9}$$

Define

$$S_m = \left\{ \lambda \in \mathbf{C}; |\operatorname{Re} \lambda - \tau_0| \leq m, |\operatorname{Im} \lambda| \geq \omega_n \right\}, \qquad m \in \mathbf{N}.$$

Relations (2.8) and (2.9) imply that for $\lambda \in S_m$ $(m \in \mathbf{N})$,

$$\| U(\tau_0 + i \operatorname{Im} \lambda, \lambda) \| \leq \tfrac{1}{2}. \tag{2.10}$$

Accordingly, (2.7) yields that $S_m \subset J$ $(m \in \mathbf{N})$ and

$$\lambda^{k-1} \overline{P_\lambda^{-1} A_k} = \lambda^{k-1} [I + U(\tau_0 + i \operatorname{Im} \lambda, \lambda)]^{-1} \overline{P_{\tau_0 + \operatorname{Im} \lambda}^{-1} A_k},$$
$$1 \leq k \leq n,\ \lambda \in S_m\ (m \in \mathbf{N}). \tag{2.11}$$

In the sequel, for convenience, we write $\lambda_0(\lambda) = \tau_0 + i \operatorname{Im} \lambda$ for $\lambda \in J$ and

$$F_k(\lambda) = \lambda^{k-1} \overline{P_\lambda^{-1} A_k}, \qquad 1 \leq k \leq n,\ \lambda \in J.$$

Then for $1 \leq k \leq n$, $\lambda \in S_m$ $(m \in \mathbf{N})$,

$$F_k(\lambda) = \left(\frac{\lambda - \lambda_0(\lambda) + \lambda_0(\lambda)}{\lambda_0(\lambda)} \right)^{k-1} [I + U(\lambda_0(\lambda), \lambda)]^{-1} F_k(\lambda_0(\lambda))$$
$$= \left\{ \sum_{i=0}^{k-1} C_{k-1}^I [\lambda_0(\lambda)]^{-i} (\lambda - \lambda_0(\lambda))^k \right\} [I + U(\lambda_0(\lambda), \lambda)]^{-1}$$
$$\cdot F_k(\lambda_0(\lambda)), \tag{2.12}$$

$$F_k^*(\lambda) = \left\{ \sum_{i=0}^{k-1} C_{k-1}^i [\bar{\lambda}_0(\lambda)]^{-i} (\lambda - \lambda_0(\lambda))^i \right\} F_k^*(\lambda_0(\lambda))$$
$$\cdot [I + U^*(\lambda_0(\lambda), \lambda)]^{-1}$$
$$= \left\{ \sum_{i=0}^{k-1} C_{k-1}^i [\bar{\lambda}_0(\lambda)]^{-i} (\lambda - \lambda_0(\lambda))^i \right\}$$
$$\cdot \left\{ F_k^*(\lambda_0(\lambda)) - F_k^*(\lambda_0(\lambda)) [I + U^*(\lambda_0(\lambda), \lambda)]^{-1} \right.$$
$$\left. \times U^*(\lambda_0(\lambda), \lambda) \right\}. \tag{2.13}$$

Moreover, by (2.1), (2.2), and (2.10), we obtain that there exists a constant $Q > 0$ such that

$$\left\| F_k^*(\lambda_0(\lambda))\left[I + U^*(\lambda_0(\lambda), \lambda)\right]^{-1} \right\| \le Q, \qquad \lambda \in S_m\ (m \in \mathbf{N}). \quad (2.14)$$

Thus, noting that

$$|\lambda_0(\lambda)|^{-i} |\lambda - \lambda_0(\lambda)|^i \le \left(\omega_m^{-1} \cdot m\right)^i \le 1, \qquad 0 \le i \le k-1,$$

we deduce that there is a constant $M > 0$ such that

$$\|F_k(\lambda)x\| \le M \|F_k(\lambda_0(\lambda))x\|, \qquad x \in H,\ \lambda \in S_m\ (m \in \mathbf{N}),\ 1 \le k \le n, \quad (2.15)$$

by looking at (2.12) and (2.10);

$$\|F_k^*(\lambda)x\| \le M \left(\sum_{i=1}^{n} |\lambda - \lambda_0(\lambda)|^i \right) \left(\sum_{j=1}^{n} \|F_j^*(\lambda_0(\lambda))x\| \right),$$
$$x \in H,\ \lambda \in S_m\ (m \in \mathbf{N}), \quad (2.16)$$

by means of (2.13), (2.14), and (2.8).

Thirdly, we show that

$$\lim_{l > m \to \infty} \int_{\tau_0 + i\omega_m}^{\tau_0 + i\omega_l} e^{\lambda t} F_k(\lambda)\, d\lambda = 0, \qquad 1 \le k \le n, \quad (2.17)$$

$$\lim_{l > m \to \infty} \int_{\tau_0 - i\omega_l}^{\tau_0 - i\omega_m} e^{\lambda t} F_k(\lambda)\, d\lambda = 0, \qquad 1 \le k \le n, \quad (2.18)$$

uniformly on compacts of $t > 0$.

In fact, a deformation of contour gives

$$\int_{\tau_0 + i\omega_m}^{\tau_0 + i\omega_l} e^{\lambda t} F_k(\lambda)\, d\lambda = \int_{\tau_0 - m + i\omega_l}^{\tau_0 + i\omega_l} e^{\lambda t} F_k(\lambda)\, d\lambda + \int_{\tau_0 + i\omega_m}^{\tau_0 - m + i\omega_m} e^{\lambda t} F_k(\lambda)\, d\lambda$$
$$+ \int_{\tau_0 - m + i\omega_m}^{\tau_0 - m + i\omega_l} e^{\lambda t} F_k(\lambda)\, d\lambda = I_1(k) + I_2(k) + I_3(k),$$
$$1 \le k \le n.$$

By (2.15), (2.1), and (2.2), for $1 \le k \le n$,

$$\|I_1(k)\| \le M \int_{\tau_0 - m}^{\tau_0} e^{\tau t} \|F_k(\tau_0 + i\omega_l)\|\, dl \le M t^{-1} e^{\tau_0 t} \|F_k(\tau_0 + i\omega_l)\|$$
$$\to 0, \qquad \text{uniformly on compacts of } t > 0, \text{ as } m \to \infty;$$
$$\|I_2(k)\| \le M t^{-1} e^{\tau_0 t} \|F_k(\tau_0 + i\omega_m)\|$$
$$\to 0, \qquad \text{uniformly on compacts of } t > 0, \text{ as } m \to \infty.$$

Integrating by parts, we obtain

$$\begin{aligned} I_3(k) &= \frac{1}{t} e^{\lambda t} F_k(\lambda)\Big|_{\tau_0 - m + i\omega_m}^{\tau_0 - m + i\omega_l} + \frac{1}{t}\int_{\tau_0 - m + i\omega_m}^{\tau_0 - m + i\omega_l} e^{\lambda t} F_k'(\lambda)\, d\lambda \\ &= I_{3,1}(k) + I_{3,2}(k), \qquad 1 \le k \le n. \end{aligned}$$

It is clear by (2.15), (2.1), and (2.2) that $\|I_{3,1}(k)\| \to 0$ $(1 \le k \le n)$ uniformly on compacts of $t > 0$, as $l > m \to \infty$. In order to estimate $I_{3,2}(k)$, we need to do some preliminary work.

First of all, by virtue of (2.15) and (2.16), we get that for any $x, y \in H$, $\lambda \in S_m$, $(m \in \mathbf{N})$, $1 \le k, j \le n$,

$$\left|\left\langle F_k(\lambda)x, (\bar{\lambda})^{-1} y\right\rangle\right| \le M\left(\tau_0^2 + \omega^2\right)^{-1/2} \|y\| \, \|F_k(\tau_0 + i\omega)x\| \tag{2.19}$$

$$\left|\left\langle F_k(\lambda)x, F_j^*(\lambda)y\right\rangle\right| \le Mnm^n \|F_k(\tau_0 + i\omega)x\| \sum_{i=1}^{n} \|F_i^*(\tau_0 + i\omega)y\|. \tag{2.20}$$

Next, (1.1), (1.3), (1.4), and Lemma 2 tell us that

$$\|F_k(\tau_0 + i\omega)x\| \in L_2(\mathbf{R}), \qquad x \in H, 1 \le k \le n. \tag{2.21}$$

Moreover, since for any $x, y \in H$, $1 \le k \le n$,

$$\left\langle y, \left[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t)\right]^* x\right\rangle = \left\langle \left[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t)\right] y, x\right\rangle, \tag{2.22}$$

we have that $[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t)]^* x$ is weakly continuous, and therefore strongly measurable. Thus for $\operatorname{Re}\lambda \ge \mu$,

$$\begin{aligned} &\left\langle \int_0^\infty e^{-\lambda t}\left[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t)\right] y\, dt, x\right\rangle \\ &\qquad = \left\langle y, \int_0^\infty e^{-\bar{\lambda} t}\left[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t)\right]^* x\, dt\right\rangle. \end{aligned}$$

According to (1.4), we obtain that for $x \in H$, $\operatorname{Re}\lambda \ge \mu$, $1 \le k \le n$,

$$F_k^*(\lambda)x = \int_0^\infty e^{-\bar{\lambda} t}\left[S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t)\right]^* x\, dt. \tag{2.23}$$

This combined with (1.1) and Lemma 2 shows that

$$\|F_k^*(\tau_0 + i\omega)x\| \in L_2(\mathbf{R}), \qquad x \in H, 1 \le k \le n. \tag{2.24}$$

Let us now estimate $I_{3,2}(k)$. In view of (2.6), for each $x, y \in H$,

$$\left|\left\langle I_{3,2}(k)x, y\right\rangle\right| = \frac{1}{t}\left|\int_{\tau_0 - m + i\omega_m}^{\tau_0 - m + i\omega_l} e^{\lambda t}\left\langle (k-1)\lambda^{-1}F_k(\lambda)x \right.\right.$$

$$\left.\left. + \sum_{j=1}^{n} jF_j(\lambda)F_k(\lambda)x, y\right\rangle d\lambda\right|$$

$$= \frac{1}{t}\left|\int_{\tau_0 - m + i\omega_m}^{\tau_0 - m + i\omega_l} e^{\lambda t}\left\{(k-1)\left\langle F_k(\lambda)x, (\bar{\lambda})^{-1}y\right\rangle\right.\right.$$

$$\left.\left. + \sum_{j=1}^{n} j\left\langle F_k(\lambda)x, F_j^*(\lambda)y\right\rangle\right\} d\lambda\right|.$$

By virtue of (2.19), (2.20), (2.21) and (2.23), we have that for any $x, y \in H$,

$$\left|\left\langle I_{3,2}(k)x, y\right\rangle\right| \leq \frac{1}{t} nm^n e^{(\tau_0 - m)t} C(x, y), \tag{2.25}$$

where $C(x, y) > 0$ depends uniquely on x and y. Accordingly, the principle of uniform boundedness yields that for each $t > 0$, there exists a constant C independent of m, l such that

$$\left\| I_{3,2}(k)\right\| \leq Ct^{-1} nm^n e^{(\tau_0 - m)t} \to 0,$$

uniformly on compacts of $t > 0$, as $l > m \to \infty$.

This completes the proof of (2.17). Reasoning in the same way with $\int_{\tau_0 - i\omega_l}^{\tau_0 - i\omega_m} e^{\lambda t} F_k(\lambda)\, d\lambda$, we obtain (2.18).

Finally, for $t > 0$, $1 \leq k \leq n-1$, set

$$T_k(t; m) = \frac{1}{2\pi i}\int_{\tau_0 - i\omega_m}^{\tau_0 + i\omega_m} e^{\lambda t} F_k(\lambda)\, d\lambda, \qquad m \in \mathbf{N}.$$

It follows from (2.17) and (2.18) that for each $t > 0$, $1 \leq k \leq n-1$, $\{T_k(t; m)\}_{m=1}^{\infty}$ is a Cauchy sequence in $L(H)$, and therefore there exist $T_k(t) \in L(H)$ $(1 \leq k \leq n-1)$ such that

$$\lim_{m \to \infty} \left\| T_k(t, m) - T_k(t)\right\| = 0, \tag{2.26}$$

uniformly on compacts of $t > 0$. Moreover, by the fact from [5, Corollary 6.4(c)] that for every $x \in \bigcap_{i=0}^{n-1} D(A_i)$ and $0 \leq k \leq n-1$, $S_k(t)x$ is the solution of (ACP_n) with $u_k^{(l)}(0) = \delta_{kl}u$, δ_{kl} the Kronecker delta $(0 \leq k, l \leq$

$n-1$), we can use [24, Lemma 3] to obtain that for $\eta > \mu$ and $x \in \bigcap_{i=0}^{n-1} D(A_i)$, $0 \le k \le n-1$,

$$S_k^{(k)}(t)x = \frac{1}{2\pi i}\int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} \sum_{i=k+1}^{n} \lambda^{i-1} \overline{P_\lambda^{-1} A_i} x \, d\lambda. \tag{2.27}$$

As a consequence, for each $0 \le k \le n-1$, $\sum_{i=k+1}^{n} T_i(t)$ coincides with $S_k^{(k)}(t)$ on the dense set $\bigcap_{k=0}^{n-1} D(A_k)$ and hence on all of H.

Obviously, for each $1 \le i \le n$, $m \in \mathbf{N}$, $T_i(t, m)$ is norm continuous for $t > 0$. According to this, (2.26) shows that for each $1 \le i \le n$, $T_i(t)$ is norm continuous for $t > 0$, and then, so does $S_k^{(k)}(t)$ $(0 \le k \le n-1)$.

Remark. Formally one can convert (ACP_n) to a first order system by introducing the auxiliary variable such as $y_0 = u$, $y_1 = u' \ldots, y_n = u^{(n-1)}$. Yet this seems to be of no particular help in proving the result above.

COROLLARY 1. *Let the characteristic condition of Theorem* 1 *be satisfied. Then for each* $0 \le k \le n-1$, $S_k(t)$ *is norm continuous for* $t > 0$.

Proof. This assertion is a direct consequence of Theorem 1 and the identity

$$S_k(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} S_k^{(k)}(s)\, ds \qquad (1 \le k \le n-1).$$

3. AN EXAMPLE

Let $a_1 > a_2 > \cdots > a_{n-1} > 0$. Suppose that $-A$ is the generator of a C_0 semigroup on H which is norm continuous for $t > 0$, with $(-\infty, 0) \subset \rho(A)$ (the resolvent set of A); $B_0, B_1, \ldots, B_{n-2}$ are closed linear operators on H such that for each $0 \le j \le n-2$, $D(B_j) \supset D(A^{(1/2)(n-j)})$, and there is a $\lambda_j \in \rho(A^{(1/2)(n-j)})$ such that $(\lambda_j - A^{(1/2)(n-j)})^{-1} B_j$ has a bounded extension. Consider the Cauchy problem

$$\begin{cases} \displaystyle\prod_{i=1}^{n-1}\left(\frac{d}{dt} + a_i A^{1/2}\right)(u'(t) + Au(t)) + \sum_{j=1}^{n-2} B_j u^{(j)}(t) = 0 \qquad (t \ge 0), \\ u^k(0) = u_k \qquad (0 \le k \le n-1). \end{cases} \tag{3.1}$$

It is known that $-A^{1/2}$ (and so for each $-a_i A^{1/2}$) generates an analytic semigroup (cf [6, Theorem 6.4.2]). Hence, there exist constants C_0, $b_0 > 0$ such that for every λ with $\operatorname{Re}\lambda > b_0$, we have $\lambda \in \rho(-a_i A^{1/2})$ and

$$\left\|\left(\lambda + a_i A^{1/2}\right)^{-1}\right\| \le C_0 |\lambda|^{-1} \qquad (\operatorname{Re}\lambda > b_0, 1 \le i \le n-1). \quad (3.2)$$

Moreover, [18, Theorem 2.3.6] gives that there is a constant $b_1 > b_0$ such that $\forall\, \tau > b_1$, $\omega \in R$, $\tau + i\omega \in \rho(-A)$ and

$$\lim_{|\omega| \to \infty} \left\|(\tau + i\omega + A)^{-1}\right\| = 0 \qquad (\forall\, \tau > b_1). \quad (3.3)$$

Noting that for $\operatorname{Re}\lambda > b_1$, $1 \le i \le j \le n-1$,

$$\begin{aligned}
&\lambda\left(\lambda + a_i A^{1/2}\right)^{-1}\left(\lambda + a_j A^{1/2}\right)^{-1} \\
&\quad = (a_i - a_j)^{-1}\left[a_i\left(\lambda + a_i A^{1/2}\right)^{-1} - a_j\left(\lambda + a_j A^{1/2}\right)^{-1}\right], \\
&A^{1/2}\left(\lambda + a_i A^{1/2}\right)^{-1}\left(\lambda + a_j A^{1/2}\right)^{-1} \\
&\quad = (a_i - a_j)^{-1}\left[\left(\lambda + a_j A^{1/2}\right)^{-1} - \left(\lambda + a_i A^{1/2}\right)^{-1}\right].
\end{aligned}$$

We get that for each $0 \le k \le n-2$, there exist constants $C_1(k), \dots, C_{n-1}(k)$ such that for $\operatorname{Re}\lambda > b_1$,

$$\lambda^k A^{(1/2)(n-k-2)} \prod_{i=1}^{n-1} \left(\lambda + a_j A^{1/2}\right)^{-1} = \sum_{i=1}^{n-1} C_i(k)\left(\lambda + a_i A^{1/2}\right)^{-1}. \quad (3.4)$$

Take $\mu_i \in \rho(a_i A^{1/2} - A)$ for each $1 \le i \le n-1$. We have that for $\operatorname{Re}\lambda > b_1$, $1 \le i \le n-1$,

$$\begin{aligned}
&\lambda\left(\lambda + a_i A^{1/2}\right)^{-1}(\lambda + A)^{-1} \\
&\quad = \left[(\mu_i + A)\left(\mu_i + A - a_i A^{1/2}\right)^{-1}\right] \\
&\qquad \times\left\{\mu_i(\lambda + \mu_i + A)^{-1}(\lambda + A)^{-1} + (\lambda + \mu_i + A)^{-1}\right\} \\
&\qquad - \left[a_i A^{1/2}\left(\mu_i + A - a_i A^{1/2}\right)^{-1}\right]\left\{\mu_i\left(\lambda + a_i A^{1/2}\right)^{-1}(\lambda + A)^{-1}\right. \\
&\qquad\qquad \left. + \left(\lambda + a_i A^{1/2}\right)^{-1}\right\}, \quad (3.5)
\end{aligned}$$

$$\begin{aligned}
A\left(\lambda + a_i A^{1/2}\right)^{-1}(\lambda + A)^{-1} &= \left(\lambda + a_i A^{1/2}\right)^{-1} \\
&\quad - \lambda\left(\lambda + a_i A^{1/2}\right)^{-1}(\lambda + A)^{-1}. \quad (3.6)
\end{aligned}$$

Now setting

$$P_0(\lambda) = (\lambda + A) \sum_{i=1}^{n-1} (\lambda + a_i A),$$

then (3.4), (3.5), and (3.6) together indicate that there exist C, $b > 0$ such that for $\operatorname{Re} \lambda > b$, $0 \le k \le n-2$, $m = 0, 1, 2, \ldots,$

$$\left\| \left[\lambda^k A^{(1/2)(n-k)} P_0^{-1}(\lambda) \right]^{(m)} \right\|, \left\| \left[\lambda^{n-1} P_0^{-1}(\lambda) \right]^{(m)} \right\|$$
$$\le Cm! \, (\operatorname{Re} \lambda - b)^{-m-1}.$$

Thus, strong wellposedness of (3.1) follows immediately from [26, 27] (or via arguing similarly as in the proof of [24, Theorem 2]). On the other hand, making use of (3.4), (3.5), and (3.6) again, we find that for $0 \le k \le n - 2$, $\tau > b_1$,

$$\lim_{|\omega| \to \infty} \left\| (\tau + i\omega)^k A^{(1/2)(n-k)} P_0^{-1}(\tau + i\omega) \right\| = 0, \tag{3.7}$$

$$\lim_{|\omega| \to \infty} \left\| (\tau + i\omega)^{n-1} P_0^{-1}(\tau + i\omega) \right\| = 0, \tag{3.8}$$

by virtue of (3.2) and (3.3).

Since, for $0 \le j \le n-1$, $\operatorname{Re} \lambda > b_1$,

$$\lambda^j \overline{P_0^{-1}(\lambda) B_j} = \lambda^j \left(\lambda_j - A^{(1/2)(n-j)} \right) P_0^{-1}(\lambda) \cdot \overline{\left(\lambda_j - A^{(1/2)(n-j)} \right)^{-1} B_j},$$

and $\overline{\left(\lambda_j - A^{(1/2)(n-j)} \right)^{-1} B_j}$ is bounded by hypothesis, we assert by (3.7) that

$$\lim_{|\omega| \to \infty} \left\| (\tau + i\omega)^j \overline{P_0^{-1}(\tau + i\omega) B_j} \right\| = 0 \qquad (\tau > b_1, 0 \le j \le n-2).$$

Accordingly, for each fixed $\tau > b_1$, when $|\omega|$ is large enough,

$$\left[I + \sum_{j=0}^{n-2} (\tau + i\omega)^j \overline{P_0^{-1}(\tau + i\omega) B_j} \right]$$

has bounded inverse. Observing

$$(\lambda + A) \prod_{i=1}^{n-1} \left(\lambda + a_i A^{1/2} \right)^{-1} + \sum_{j=0}^{n-2} B_j \lambda^j = P_0(\lambda) \left(I + \sum_{j=0}^{n-2} \lambda^j P_0^{-1}(\lambda) B_j \right),$$

we conclude by (3.7) and (3.8) that the hypothesis in Theorem 1 is satisfied. Therefore, the propagators $S_0(t), S_1(t), \ldots, S_{n-1}(t)$ of (3.1), as well as $S_k^{(k)}(t)$ ($\forall\ 1 \le k \le n-1$), are norm continuous for $t > 0$.

ACKNOWLEDGMENT

The authors thank Professor E. B. Davies for his useful suggestions. The authors also thank the referee for his helpful comments.

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